Separable Equations

Def: any first order ODE that can be written in the form

$$g(y)\frac{dy}{dx} = f(x)$$

or $g(y)\,dy = f(x)\,dx$

is separable and can be solved by integration.

If $g(y)\frac{dy}{dx} = f(x)$

then $\int g(y(x))\frac{dy}{dx}\,dx = \int f(x)\,dx$

or $\int g(y)\,dy = \int f(x)\,dx$.

For this reason you frequently see

$$g(y)\,dy = f(x)\,dx$$

$$\int g(y)\,dy = \int f(x)\,dx$$

just make sure you understand that, technically, you are integrating both sides of the equation $g(y)\frac{dy}{dx} = f(x)$ with respect to x.

ex: if $\frac{dy}{dx} = f(x)$ then the equation is always separable.

$$\int \frac{dy}{dx}\,dx = \int f(x)\,dx$$

$$\int dy = \int f(x)\,dx$$

and $y = \int f(x)\,dx$

As an example, find the general solution of

$$\frac{dy}{dx} = 3x^2$$

integrating w.r.t. x we have

$$\int \frac{dy}{dx}\,dx = \int 3x^2\,dx$$

$$\boxed{y = x^3 + C}$$

you will often see

$$dy = 3x^2\,dx$$

$$\int dy = \int 3x^2\,dx$$

$\boxed{y = x^3 + 3}$ Make sure you understand why this is true.

ex: Find the general solution of

$$xy(1+y^2)dx - (1+x^2)dy = 0$$

This equation is separable (we can get x on one side, y on the other)

$$\int \frac{x}{1+x^2}dx = \int \frac{dy}{y(1+y^2)}$$

The left side is a simple substitution, $u = 1+x^2$. The right side is partial fractions –

$$\frac{1}{y(1+y^2)} = \frac{A}{y} + \frac{By+C}{1+y^2}$$

$$1 = A + Ay^2 + By^2 + Cy$$

so $A = 1$, $C = 0$ and $A+B = 0$ so $B = -1$

Thus $\displaystyle\int \frac{x\,dx}{1+x^2} = \int \left\{ \frac{1}{y} - \frac{y}{1+y^2} \right\} dy$

$$\frac{1}{2}\ln(1+x^2) = \ln|y| - \frac{1}{2}\ln(1+y^2) + c$$

$$\frac{1}{2}\ln(1+x^2) = \frac{1}{2}\ln(y^2) - \frac{1}{2}\ln(1+y^2) + c$$

$$\ln(1+x^2) = \ln\left(\frac{y^2}{1+y^2}\right) + c \quad \text{where } 2c = c.$$

$$\ln(1+x^2) - \ln\left(\frac{y^2}{1+y^2}\right) = c$$

$$\boxed{\frac{(1+x^2)(1+y^2)}{y^2} = C} \quad \text{where } e^c = C$$

is the general solution.

I might write this as

$\boxed{(1+x^2)(1+y^2) = Cy^2}$ because it looks nicer

but you can not write $y(x)$ as one explicit function of x so, however you write it, your answer will be implicit.

Ex: Find the general solution of the IVP

$$\frac{dy}{dx} = \frac{e^x}{y}, \quad y(0) = 1$$

This equation is separable –

$$y\,dy = e^x\,dx$$

integrating, we have -

$$\int y\,dy = \int e^x dx$$

$$\frac{y^2}{2} = e^x + c$$

Now apply the condition $y(0) = 1$

$$\frac{1}{2} = e^0 + c$$

$$c = -\frac{1}{2}$$

and $\frac{y^2}{2} = e^x - \frac{1}{2}$

or $y^2 = 2e^x - 1$

We know there is exactly one solution but if we solve for y explicitly we find two solutions

$$y = \pm\sqrt{2e^x - 1}$$

But notice that only one of these solutions satisfies the initial condition, that is

$$\sqrt{2e^0 - 1} = \sqrt{1} = 1$$

But $-\sqrt{2e^0 - 1} = -\sqrt{1} = -1 \neq 1$

So the second expression is not a solution and

$\boxed{y = \sqrt{2e^x - 1}}$ is the solution of the IVP.

The most important separable equation -

The most prevalent 1^{st} order ODE in application is the equation $\frac{dx}{dt} = kx$.

This equation says that the rate of change of the quantity x is proportional to the amount of x present.

If $k > 0$ then $\frac{dx}{dt}$ is positive and the equation describes growth of x. If $k < 0$ then $\frac{dx}{dt}$ is negative and the equation describes decay.

Some applications include radioactive decay, reaction rate, population growth, and interest bearing accounts.

Everyone on earth should know how to solve this equation. It is separable —

$$\frac{dx}{dt} = kx$$

$$\frac{dx}{x} = k\,dt$$

$$\int \frac{dx}{x} = \int k\,dt$$

$$\ln|x| = kt + c$$

$$|x| = e^{kt+c}$$

$$= e^{c} e^{kt}$$

$$|x| = Ce^{kt} \quad \text{where } e^{c} = C$$

If we assume that $t \geq 0$ and $x \geq 0$ in real physical situations the solution becomes

$$x(t) = Ce^{kt}$$

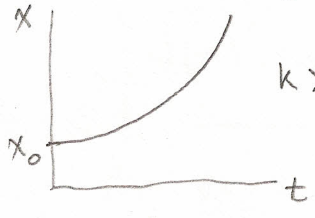
Furthermore, if we apply the usual condition that $x(0) = x_0$

then $x(0) = x_0 = Ce^{0} = C$

so $C = x_0$ and

$$x(t) = x_0 e^{kt}$$ where x_0 is the quantity of x present at $t = 0$.

k is called the growth constant if $k > 0$ and the decay constant if $k < 0$.



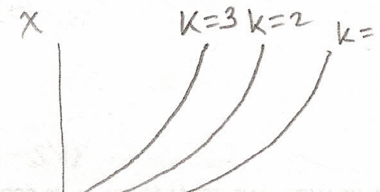
$k > 0$, growth

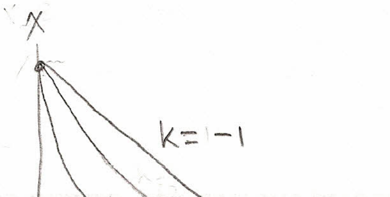
$x \to \infty$ as $t \to \infty$

$k < 0$, decay

$x \to 0$ as $t \to \infty$

Notice also that the magnitude of k determines the rate of increase or decrease of x. Large k is fast growth or decay, small k is slow.





Ex: The number of bacteria $N(t)$ in a sample is observed to double every 8 seconds. Find an expression for $N(t)$ if $N(0) = 100$.

We know the rate of growth is proportional to the amount present, that is –

$$\frac{dN}{dt} = kN$$

$$\frac{dN}{N} = k\,dt$$

$$\ln N = kt + C$$

$$N = Ce^{kt}$$

We actually have 2 unknowns here, C and k.

We can find C by applying the initial condition $N(0) = 100$ –

$$100 = Ce^{0} = C$$

So $N(t) = 100e^{kt}$

We need to determine the growth constant k. We know that N doubles every 8 seconds so

$N(8) = 200$ and

$$200 = 100e^{8k}$$

and $k = \frac{\ln 2}{8} \approx 0.086643$

So the solution is given by

$$\boxed{N(t) = 100e^{0.086643t}}$$

And you can check by verifying that

$N(8) = 200$

$N(16) = 400$

$N(24) = 800$

$\vdots$

etc.